Spectral study of alliances in graphs

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Abstract

In this paper we obtain several tight bounds on different types of alliance numbers of a graph, namely (global) defensive alliance number, global offensive alliance number and global dual alliance number. In particular, we investigate the relationship between the alliance numbers of a graph and its algebraic connectivity, its spectral radius, and its Laplacian spectral radius.

Keywords: Defensive alliance, offensive alliance, dual alliance, domination, spectral radius, graph eigenvalues.

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1 Introduction

The study of defensive alliances in graphs, together with a variety of other kinds of alliances, was introduced by Hedetniemi, et. al. [2]. In the referred paper was initiated the study of the mathematical properties of alliances.

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In particular, several bounds on the defensive alliance number were given. The particular case of global (strong) defensive alliance was investigated in[3] where several bounds on the global (strong) defensive alliance number were obtained.

In this paper we obtain several tight bounds on different types of alliance numbers of a graph, namely (global) defensive alliance number, global offensive alliance number and global dual alliance number. In particular, we investigate the relationship between the alliance numbers of a graph and its algebraic connectivity, its spectral radius, and its Laplacian spectral radius.

We begin by stating some notation and terminology. In this paper $\Gamma = (V, E)$ denotes a simple graph of order n and size m. For a non-empty subset $S \subseteq V$, and any vertex $v \in V$, we denote by $N_S(v)$ the set of neighbors v has in S:

$$N_S(v) := \{u \in S : u \sim v\},$$

Similarly, we denote by $N_{V\setminus S}(v)$ the set of neighbors v has in $V \setminus S$:

$$N_{V\setminus S}(v) := \{u \in V \setminus S : u \sim v\}.$$

In this paper we will use the following obvious but useful claims:

Claim 1. *Let $\Gamma = (V, E)$ be a simple graph of size m. If $S \subset V$, then*

$$2m = \sum_{v\in S} |N_S(v)| + 2\sum_{v\in S} |N_{V\setminus S}(v)| + \sum_{v\in V\setminus S} |N_{V\setminus S}(v)|.$$

Claim 2. *Let $\Gamma = (V, E)$ be a simple graph. If $S \subset V$, then*

$$\sum_{v\in S} |N_{V\setminus S}(v)| = \sum_{v\in V\setminus S} |N_S(v)|.$$

Claim 3. *Let $\Gamma = (V, E)$ be a simple graph. If $S \subset V$, then*

$$\sum_{v\in S} |N_S(v)| \le |S|(|S| - 1).$$

2 Defensive alliances

A nonempty set of vertices $S \subseteq V$ is called a *defensive alliance* if for every $v \in S$,

$$|N_S(v)| + 1 \geq |N_{V \setminus S}(v)|.$$

In this case, by strength of numbers, every vertex in S is *defended* from possible attack by vertices in $V \setminus S$. A defensive alliance S is called *strong* if for every $v \in S$,

$$|N_S(v)| \geq |N_{V \setminus S}(v)|.$$

In this case every vertex in S is *strongly defended*.

The *defensive alliance number* $a(\Gamma)$ (respectively, *strong defensive alliance number* $\hat{a}(\Gamma)$) is the minimum cardinality of any defensive alliance (respectively, strong defensive alliance) in Γ.

A particular case of alliance, called global defensive alliance, was studied in [3]. A defensive alliance S is called *global* if it affects every vertex in $V \setminus S$, that is, every vertex in $V \setminus S$ is adjacent to at least one member of the alliance S. Note that, in this case, S is a dominating set. The *global defensive alliance number* $\gamma_a(\Gamma)$ (respectively, *global strong defensive alliance number* $\gamma_{\hat{a}}(\Gamma)$) is the minimum cardinality of any global defensive alliance (respectively, global strong defensive alliance) in Γ.

2.1 Algebraic connectivity and defensive alliances

It is well-known that the second smallest Laplacian eigenvalue of a graph is probably the most important information contained in the Laplacian spectrum. This eigenvalue, frequently called *algebraic connectivity*, is related to several important graph invariants and imposes reasonably good bounds on the values of several parameters of graphs which are very hard to compute.

The algebraic connectivity of Γ, μ, satisfies the following equality showed by Fiedler [1] on weighted graphs

$$\mu = 2n \min \left\{ \frac{\sum_{v_i \sim v_j} (w_i - w_j)^2}{\sum_{v_i \in V} \sum_{v_j \in V} (w_i - w_j)^2} : w \neq \alpha \mathbf{j} \text{ for } \alpha \in \mathbb{R} \right\}, \qquad (1)$$

where $V = \{v_1, v_2, ..., v_n\}$, $\mathbf{j} = (1, 1, ..., 1)$ and $w \in \mathbb{R}^n$.

The following theorem shows the relationship between the algebraic connectivity of a graph and its (strong) defensive alliance number.

Theorem 4. *Let Γ be a simple graph of order n. Let μ be the algebraic connectivity of Γ. The defensive alliance number of Γ is bounded by*

$$a(\Gamma) \geq \left\lceil \frac{n\mu}{n+\mu} \right\rceil$$

and the strong defensive alliance number of Γ is bounded by

$$\hat{a}(\Gamma) \geq \left\lceil \frac{n(\mu+1)}{n+\mu} \right\rceil.$$

Proof. If S denotes a defensive alliance in Γ, then

$$|N_{V\setminus S}(v)| \leq |S|, \quad \forall v \in S. \tag{2}$$

From (1), taking $w \in \mathbb{R}^n$ defined as

$$w_i = \begin{cases} 1 & \text{if} \quad v_i \in S; \\ 0 & \text{otherwise,} \end{cases}$$

we obtain

$$\mu \leq \frac{n \sum_{v\in S} |N_{V\setminus S}(v)|}{|S|(n-|S|)}. \tag{3}$$

Thus, (2) and (3) lead to

$$\mu \leq \frac{n|S|}{n-|S|}. \tag{4}$$

Therefore, solving (4) for $|S|$, and considering that it is an integer, we obtain the bound on $a(\Gamma)$. Moreover, if the defensive alliance S is strong, then by (3) and Claim 3 we obtain

$$\mu \leq \frac{n \sum_{v\in S} |N_S(v)|}{|S|(n-|S|)} \leq \frac{n(|S|-1)}{n-|S|}. \tag{5}$$

Hence, the result follows. □

The above bounds are sharp as we can check in the following examples. It was shown in [2] that, for the complete graph $\Gamma = K_n$, $a(K_n) = \lceil \frac{n}{2} \rceil$ and $\hat{a}(K_n) = \lceil \frac{n+1}{2} \rceil$. As the algebraic connectivity of K_n is $\mu = n$, the above theorem gives the exact value of $a(K_n)$ and $\hat{a}(K_n)$. Moreover, if Γ is the icosahedron, then $a(\Gamma) = 3$. Since in this case $n = 12$ and $\mu = 5 - \sqrt{5}$, the above theorem gives $a(\Gamma) \geq 3$.

Theorem 5. *Let Γ be a simple and connected graph of order n and maximum degree Δ. Let μ be the algebraic connectivity of Γ. The strong defensive alliance number of Γ is bounded by*

$$\hat{a}(\Gamma) \geq \left\lceil \frac{n(\mu - \lfloor \frac{\Delta}{2} \rfloor)}{\mu} \right\rceil.$$

Proof. If S denotes a strong defensive alliance in Γ, then

$$|N_{V\setminus S}(v)| \leq \left\lfloor \frac{deg(v)}{2} \right\rfloor \quad \forall v \in S. \tag{6}$$

Thus, by (3) the result follows. □

The bound is attained, for instance, in the the following cases: the complete graph $\Gamma = K_n$, the Petersen graph, and the 3-cube graph.

2.2 Bounds on the global defensive alliance number

The spectral radius of a graph is the largest eigenvalue of its adjacency matrix. It is well-known that the spectral radius of a graph is directly related with several parameters of the graph. The following theorem shows the relationship between the spectral radius of a graph and its global (strong) defensive alliance number.

Theorem 6. *Let Γ be a simple graph of order n. Let λ be the spectral radius of Γ. The global defensive alliance number of Γ is bounded by*

$$\gamma_a(\Gamma) \geq \left\lceil \frac{n}{\lambda + 2} \right\rceil$$

and the global strong defensive alliance number of Γ is bounded by

$$\gamma_{\hat{a}}(\Gamma) \geq \left\lceil \frac{n}{\lambda + 1} \right\rceil.$$

Proof. If S denotes a defensive alliance in Γ, then

$$\sum_{v\in S} |N_{V\setminus S}(v)| \leq \sum_{v\in S} |N_S(v)| + |S|. \tag{7}$$

Moreover, if the defensive alliance S is global, we have

$$n - |S| \leq \sum_{v \in S} |N_{V \setminus S}(v)|. \tag{8}$$

Thus, by (7) and (8) we obtain

$$n - 2|S| \leq \sum_{v \in S} |N_S(v)|. \tag{9}$$

On the other hand, if $\mathbf{A}$ denotes the adjacency matrix if Γ, we have

$$\frac{\langle \mathbf{A}w, w\rangle}{\langle w, w\rangle} \leq \lambda, \quad \forall w \in \mathbb{R}^n \setminus \{0\}. \tag{10}$$

Thus, taking w as in the proof of Theorem 4, we obtain

$$\sum_{v \in S} |N_S(v)| \leq \lambda |S|. \tag{11}$$

By (9) and (11), considering that $|S|$ is an integer, we obtain the bound on $\gamma_a(\Gamma)$. Moreover, if the defensive alliance S is strong , then

$$\sum_{v \in S} |N_{V \setminus S}(v)| \leq \sum_{v \in S} |N_S(v)|. \tag{12}$$

Thus, by (8), (12) and (11), we obtain $n - |S| \leq \lambda |S|$. Hence, the result follows. $\square$

To show the tightness of above bounds we consider, for instance, the graph $\Gamma = P_2 \times P_3$ and the graph of Figure 1. The spectral radius of $P_2 \times P_3$ is $\lambda = 1 + \sqrt{2}$, then we have $\gamma_a(\Gamma) \geq 2$. The spectral radius of the graph of Figure 1 is $\lambda = 3$, then the above theorem leads to $\gamma_{\hat{a}}(\Gamma) \geq 3$. Hence, the bounds are tight.

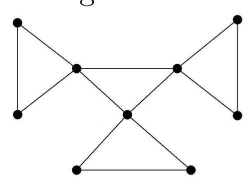
It was shown in [3] that if Γ has maximum degree Δ, its global defensive alliance number is bounded by

$$\gamma_a(\Gamma) \geq \frac{n}{\lceil \frac{\Delta}{2} \rceil + 1} \tag{13}$$

and its global strong defensive alliance number is bounded by

$$\gamma_{\hat{a}}(\Gamma) \geq \sqrt{n}. \tag{14}$$

Figure 1:



Moreover, it was shown in [3] that if Γ is bipartite, then its global defensive alliance number is bounded by

$$\gamma_a(\Gamma) \geq \left\lceil \frac{2n}{\Delta+3} \right\rceil . \tag{15}$$

The following result shows that the bound (15) is not restrictive to the case of bipartite graphs. Moreover, we obtain a bound on $\gamma_{\hat{a}}$ that improves the bound (14) in the cases of graphs of order n such that $n > \left(\left\lfloor \frac{\Delta}{2} \right\rfloor + 1\right)^2$.

Theorem 7. *Let Γ be a simple graph of order n and maximum degree Δ. The global defensive alliance number of Γ is bounded by*

$$\gamma_a(\Gamma) \geq \left\lceil \frac{2n}{\Delta+3} \right\rceil$$

and then global strong defensive alliance number of Γ is bounded by

$$\gamma_{\hat{a}}(\Gamma) \geq \left\lceil \frac{n}{\left\lfloor \frac{\Delta}{2} \right\rfloor + 1} \right\rceil .$$

Proof. If S denotes a global defensive alliance in Γ, then by (8) and (9) we have

$$2n - 3|S| \leq \sum_{v \in S} \left(|N_{V \setminus S}(v)| + |N_S(v)|\right) = \sum_{v \in S} deg(v) \leq |S|\Delta. \tag{16}$$

Thus, the bound on $\gamma_a(\Gamma)$ follows. Moreover, if the strong defensive alliance S is global, by (8) and (6) we obtain $n \leq |S| \left(1 + \left\lfloor \frac{\Delta}{2} \right\rfloor\right)$. Hence, the bound on $\gamma_{\hat{a}}(\Gamma)$ follows. $\square$

The tightness of the above bound of $\gamma_a(\Gamma)$ was showed in [3] for the case of bipartite graphs. Moreover, the above bound of $\gamma_{\hat{a}}(\Gamma)$ is attained, for instance, in the case of the Petersen graph.

2.3 The girth of regular graphs of small degree

The length of a smallest cycle in a graph Γ is called the *girth* of Γ, and is denoted by $girth(\Gamma)$. It was shown in [2] that,

(i) if Γ is regular of degree $\delta = 3$ or $\delta = 4$, then $\hat{a}(\Gamma) = girth(\Gamma)$,

(ii) if Γ is 5-regular, then $a(\Gamma) = girth(\Gamma)$.

As a consequence of the previous results we obtain interesting relations between the girth and the algebraic connectivity of regular graphs with small degree.

Theorem 8. *Let Γ be a simple and connected graph of order n. Let μ be the algebraic connectivity of Γ. Then,*

- *if Γ is 3-regular, then $girth(\Gamma) \geq \left\lceil \frac{n(\mu-1)}{\mu} \right\rceil$;*
- *if Γ is 4-regular, then $girth(\Gamma) \geq \left\lceil \frac{n(\mu-2)}{\mu} \right\rceil$;*
- *if Γ is 5-regular, then $girth(\Gamma) \geq \left\lceil \frac{n\mu}{n+\mu} \right\rceil$.*

Proof. The results are direct consequence of (i), (ii), Theorem 5 and Theorem 4. □

In order to show the effectiveness of above bounds we consider the following examples in which the bounds lead to the exact values of the girth. If Γ is the Petersen graph, $\delta = 3$, $n = 10$ and $\mu = 2$, then we have $girth(\Gamma) \geq 5$. If $\Gamma = K_6 - F$, where F is a 1-factor, $\delta = 4$, $n = 6$ and $\mu = 4$, then we have $girth(\Gamma) \geq 3$. If Γ is the icosahedron, $\delta = 5$, $n = 12$ and $\mu = 5 - \sqrt{5}$, then we have $girth(\Gamma) \geq 3$.

3 Offensive alliances

The boundary of a set $S \subset V$ is defined as

$$\partial(S) := \bigcup_{v \in S} N_{V \setminus S}(v).$$

A non-empty set of vertices $S \subseteq V$ is called *offensive alliance* if and only if for every $v \in \partial(S)$,

$$|N_S(v)| \geq |N_{V\setminus S}(v)| + 1.$$

An offensive alliance S is called *strong* if for every vertex $v \in \partial(S)$,

$$|N_S(v)| \geq |N_{V\setminus S}(v)| + 2.$$

A non-empty set of vertices $S \subseteq V$ is a *global offensive alliance* if for every vertex $v \in V \setminus S$,

$$|N_S(v)| \geq |N_{V\setminus S}(v)| + 1.$$

Thus, global offensive alliances are also dominating sets, and one can define the *global offensive alliance number*, denoted $\gamma_{a_o}(\Gamma)$, to equal the minimum cardinality of a global offensive alliance in Γ. Analogously, $S \subseteq V$ is a *global strong offensive alliance* if for every vertex $v \in V \setminus S$,

$$|N_S(v)| \geq |N_{V\setminus S}(v)| + 2,$$

and the *global strong offensive alliance number*, denoted $\gamma_{\hat{a}_o}(\Gamma)$, is defined as the minimum cardinality of a global strong offensive alliance in Γ.

3.1 Bounds on the global offensive alliance number

Similarly to (1), the Laplacian spectral radius of Γ (the largest Laplacian eigenvalue of Γ), μ_*, satisfies

$$\mu_* = 2n \max\left\{ \frac{\sum_{v_i \sim v_j}(w_i - w_j)^2}{\sum_{v_i\in V}\sum_{v_j\in V}(w_i - w_j)^2} : w \neq \alpha\mathbf{j} \text{ for } \alpha \in \mathbb{R} \right\}. \tag{17}$$

The following theorem shows the relationship between the Laplacian spectral radius of a graph and its global (strong) offensive alliance number.

Theorem 9. *Let Γ be a simple graph of order n and minimum degree δ. Let μ_* be the Laplacian spectral radius of Γ. The global offensive alliance number of Γ is bounded by*

$$\gamma_{a_o}(\Gamma) \geq \left\lceil \frac{n}{\mu_*}\left\lceil \frac{\delta+1}{2}\right\rceil \right\rceil$$

and the global strong offensive alliance number of Γ is bounded by

$$\gamma_{\hat{a}_o}(\Gamma) \geq \left\lceil \frac{n}{\mu_*}\left(\left\lceil \frac{\delta}{2}\right\rceil + 1\right) \right\rceil.$$

Proof. Let $S \subseteq V$. By (17), taking $w \in \mathbb{R}^n$ as in the proof of Theorem 4 we obtain

$$\mu_* \geq \frac{n \sum_{v \in V \setminus S} |N_S(v)|}{|S|(n - |S|)}. \tag{18}$$

Moreover, if S is a global offensive alliance in Γ,

$$|N_S(v)| \geq \left\lceil \frac{deg(v) + 1}{2} \right\rceil \quad \forall v \in V \setminus S. \tag{19}$$

Thus, (18) and (19) lead to

$$\mu_* \geq \frac{n}{|s|} \left\lceil \frac{\delta + 1}{2} \right\rceil. \tag{20}$$

Therefore, solving (20) for $|S|$, and considering that it is an integer, we obtain the bound on $\gamma_{a_o}(\Gamma)$. If the global offensive alliance S is strong, then

$$|N_S(v)| \geq \left\lceil \frac{deg(v)}{2} \right\rceil + 1 \quad \forall v \in V \setminus S. \tag{21}$$

Thus, (18) and (21) lead to the bound on $\gamma_{\hat{a}_o}(\Gamma)$. $\square$

If Γ is the Petersen graph, then $\mu_* = 5$. Thus, Theorem 9 leads to $\gamma_{a_o}(\Gamma) \geq 4$ and $\gamma_{\hat{a}_o}(\Gamma) \geq 6$. Therefore, the above bounds are tight.

Theorem 10. *Let Γ be a simple graph of order n, size m and maximum degree Δ. The global offensive alliance number of Γ is bounded by*

$$\gamma_{a_0}(\Gamma) \geq \left\lceil \frac{(2n + \Delta + 1) - \sqrt{(2n + \Delta + 1)^2 - 8(2m + n)}}{4} \right\rceil$$

and the global strong offensive alliance number of Γ is bounded by

$$\gamma_{\hat{a_0}}(\Gamma) \geq \left\lceil \frac{(2n + \Delta + 2) - \sqrt{(2n + \Delta + 2)^2 - 16(m + n)}}{4} \right\rceil.$$

Proof. If S is a global offensive alliance in $\Gamma = (V, E)$, then

$$\sum_{v \in V \setminus S} |N_S(v)| \geq \sum_{v \in V \setminus S} |N_{V \setminus S}(v)| + (n - |S|). \tag{22}$$

Moreover,

$$|S|(n-|S|) \geq \sum_{v \in V \setminus S} |N_S(v)|. \tag{23}$$

Hence,

$$(|S|-1)(n-|S|) \geq \sum_{v \in V \setminus S} |N_{V \setminus S}(v)|. \tag{24}$$

Thus,

$$(2|S|-1)(n-|S|) \geq \sum_{v \in V \setminus S} |N_S(v)| + \sum_{v \in V \setminus S} |N_{V \setminus S}(v)| = \sum_{v \in V \setminus S} deg(v). \tag{25}$$

Therefore,

$$(2|S|-1)(n-|S|) + \Delta|S| \geq \sum_{v \in V \setminus S} deg(v) + \sum_{v \in S} deg(v) = 2m. \tag{26}$$

Thus, the bound on $\gamma_{a_0}(\Gamma)$ follows. If the global offensive alliance S is strong, then we have

$$\sum_{v \in V \setminus S} |N_S(v)| \geq \sum_{v \in V \setminus S} |N_{V \setminus S}(v)| + 2(n-|S|). \tag{27}$$

Basically the bound on $\gamma_{\hat{a_0}}(\Gamma)$ follows as before: by replacing (22) by (27). $\square$

The above bounds are tight as we can see, for instance, in the case of the complete graph $\Gamma = K_n$ and the complete bipartite graph $\Gamma = K_{3,6}$, for the bound on $\gamma_{a_0}(\Gamma)$, and in the case of the complete bipartite graph $\Gamma = K_{3,3}$, for the bound on $\gamma_{\hat{a_0}}(\Gamma)$.

4 Dual alliances

An alliance is called *dual* if it is both defensive and offensive. The *global dual alliance number* of a graph Γ, denoted by $\gamma_{a_d}(\Gamma)$, is defined as the minimum cardinality of any global dual alliance in Γ. In the case of *strong* alliances we denote the global dual alliance number by $\gamma_{\hat{a_d}}(\Gamma)$.

4.1 Bounds on the global dual alliance number

Theorem 11. *Let Γ be a simple graph of order n and size m. Let λ be the spectral radius of Γ. The global dual alliance number is of Γ is bounded by*

$$\gamma_{a_d}(\Gamma) \geq \left\lceil \frac{2m+n}{4(\lambda+1)} \right\rceil$$

and the global strong dual alliance number is of Γ is bounded by

$$\gamma_{\hat{a}_d}(\Gamma) \geq \left\lceil \frac{m+n}{2\lambda+1} \right\rceil .$$

Proof. Let S be a global dual alliance in $\Gamma = (V, E)$. Since S is a global offensive alliance, S satisfies (22). Hence, by (22) and Claim 1 we obtain

$$\sum_{v\in V\setminus S} |N_S(v)| \geq \left(2m - \sum_{v\in S} |N_S(v)| - 2\sum_{v\in S} |N_{V\setminus S}(v)|\right) + n - |S|$$

Moreover, since the alliance S is defensive, by (7) and by Claim 2 we have

$$4|s| + 4\sum_{v\in S} |N_S(v)| \geq 2m + n. \tag{28}$$

Hence, by (11), the bound on $\gamma_{a_d}(\Gamma)$ follows. On the other hand, if the global offensive alliance S is strong, then

$$\sum_{v\in V\setminus S} |N_S(v)| \geq \sum_{v\in V\setminus S} |N_{V\setminus S}(v)| + 2(n - |S|).$$

Hence, by Claim 1 we have

$$\sum_{v\in V\setminus S} |N_S(v)| \geq \left(2m - \sum_{v\in S} |N_S(v)| - 2\sum_{v\in S} |N_{V\setminus S}(v)|\right) + 2(n - |S|).$$

and by Claim 2 we have

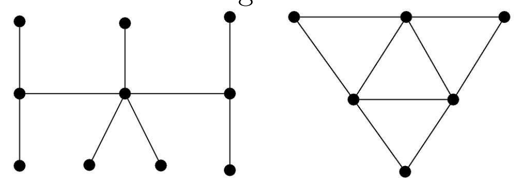
$$\sum_{v\in S} |N_S(v)| + 3\sum_{v\in S} |N_{V\setminus S}(v)| \geq 2m + 2(n - |S|).$$

Moreover, as the strong alliance S is defensive, by (12) we have

$$2\sum_{v\in S} |N_S(v)| \geq m + n - |S|. \tag{29}$$

Hence, by (11), the bound on $\gamma_{\hat{a}_d}(\Gamma)$ follows. ∎

Figure 2:



For the left hand side graph of Figure 2 we have $\lambda = \sqrt{6}$. Thus, Theorem 11 leads to $\gamma_{a_d}(\Gamma) \geq 3$. Moreover, for the right hand side graph of Figure 2 we have $\lambda = 1 + \sqrt{5}$. Thus, Theorem 11 leads to $\gamma_{\hat{a_d}}(\Gamma) \geq 3$. Hence, the above bounds are attained.

Theorem 12. *Let Γ be a simple graph of order n and size m. The global dual alliance number is of Γ is bounded by*

$$\gamma_{a_d}(\Gamma) \geq \left\lceil \frac{\sqrt{2m+n}}{2} \right\rceil$$

and the global strong dual alliance number is of Γ is bounded by

$$\gamma_{\hat{a_d}}(\Gamma) \geq \left\lceil \frac{1+\sqrt{1+8(n+m)}}{4} \right\rceil.$$

Proof. Let S be a global dual alliance in $\Gamma = (V, E)$. By (28) and Claim 3 we obtain the bound on $\gamma_{a_d}(\Gamma)$. On the other hand, if the alliance S is strong, by (29) and Claim 3 we obtain the bound on $\gamma_{\hat{a_d}}(\Gamma)$. $\square$

The above bounds are tight as we can see, for instance, in the case of the complete graph $\Gamma = K_n$, for the bound on $\gamma_{a_d}(\Gamma)$, and $\Gamma = K_1 * (K_2 \cup K_2)$, for the bound on $\gamma_{\hat{a_d}}(\Gamma)$, where $K_1 * (K_2 \cup K_2)$ denotes the joint of the trivial graph K_1 and the graph $K_2 \cup K_2$ (obtained from K_1 and $K_2 \cup K_2$ by joining the vertex of K_1 with every vertex of $K_2 \cup K_2$). Moreover, both bounds are attained in the case of the right hand side graph of Figure 2.

5 Additional observations

By definition of global alliance, any global (defensive or offensive) alliance is a dominating set. The *domination number* of a graph Γ, denoted by $\gamma(\Gamma)$,

is the size of its smallest dominating set(s). Therefore, $\gamma_a(\Gamma) \geq \gamma(\Gamma)$ and $\gamma_{a_o}(\Gamma) \geq \gamma(\Gamma)$. It was shown in [4] (for the general case of hypergraphs) that

$$\gamma(\Gamma) \geq \frac{n}{\mu_*},$$

where μ_* denotes the Laplacian spectral radius of Γ.

The reader interested in the particular case of global alliances in planar graphs is referred to [5] for a detailed study.

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